§ Integral Calculus on Surfaces (do Carmo § 2.5)

We have talked about differential calculus on surfaces.

Now, we move on to integration.

Question: Given a function $f : S \to \mathbb{R}$ on a surface S,

how to define $\int_S f$?

Some properties about integration on $\mathbb{R}^2$:

(1) For any bounded subset $\mathcal{U} \subseteq \mathbb{R}^2$,

$$\int_{\mathcal{U}} 1 = \text{Area}(\mathcal{U})$$

(2) Change of variable formula:

$$\boxed{\int_{\mathcal{U}} f(x,y)\, dx\, dy = \int_{\mathcal{U}'} f(u,v)\, |\text{Jac}\, \phi|\, du\, dv}$$

where $\phi : \mathcal{U}' \to \mathcal{U}$ is the change of coordinate transformation

$$\phi(u,v) = (x(u,v), y(u,v))$$

with Jacobian determinant

$$\text{Jac}\, \phi := \det(d\phi) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

E.g. $|\text{Jac}\, \phi| = r$ for polar coordinates r, θ

(3) $\exists$ various "Fundamental Theorems of Calculus"

Green's Theorem, Stokes' Theorem, Divergence Theorem

$$\boxed{\int_{\Omega} \text{"}d\text{"}\omega = \int_{\partial\Omega} \omega}$$

Def$^{\underline{n}}$: The support of a function $f: S \to \mathbb{R}$ is defined as

$$\operatorname{spt}(f) := \overline{\{x \in S : f(x) \neq 0\}}$$

closure in S

Def$^{\underline{n}}$: Let $\Phi: \mathcal{U} \xrightarrow{\cong} V \subseteq S$ be a parametrization of S

and $f: S \to \mathbb{R}$ be a function s.t. $\operatorname{spt}(f) \subseteq V$

Define

$$(*) \cdots \quad \int_S f := \int_{\mathcal{U}} f \circ \Phi \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| du\, dv$$

Remark: The def$^{\underline{n}}$ is independent of the choice of Φ, i.e.

if $\Phi': \mathcal{U}' \xrightarrow{\cong} V$ is another parametrization, then

$$\int_{\mathcal{U}} f \circ \Phi \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| du\, dv = \int_{\mathcal{U}'} f \circ \Phi' \left\| \frac{\partial \Phi'}{\partial u'} \times \frac{\partial \Phi'}{\partial v'} \right\| du'\, dv'$$

$\because$ change of variable formula (Ex: Prove this!)

Note: If the function f is not supported on a single coordinate neighborhood, i.e. $\text{spt}(f) \nsubseteq V$, one can use a "partition of unity" to decompose into a (finite) sum

$$f = \sum_\alpha f_\alpha \quad \text{s.t.} \quad \text{spt}(f_\alpha) \subseteq V_\alpha$$

s.t. each f_α is contained in a single coord. nbd. V_α. Then,

$$\int_S f := \sum_\alpha \int_S f_\alpha$$

In practice, if $X : \mathcal{U} \to S$ is a parametrization which covers almost all of S (except a set of "measure zero") then $(*)$ is still applicable.

Example: (Area of the sphere)

The parametrization $X : (0, 2\pi) \times (0, \pi) \longrightarrow \mathbb{S}^2(r)$

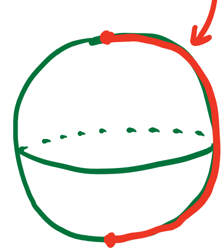
$$X(\theta, \varphi) := (r \sin\varphi \cos\theta, \ r \sin\varphi \sin\theta, \ r\cos\varphi)$$

covers almost the whole sphere $\mathbb{S}^2(r)$ except for a latitude

$$\text{Area}(\mathbb{S}^2(r)) = \int_{\mathbb{S}^2(r)} 1$$

$$= \int_0^\pi \int_0^{2\pi} 1 \cdot \underbrace{r^2 \sin\varphi}_{\left\| \frac{\partial X}{\partial \theta} \times \frac{\partial X}{\partial \varphi} \right\|} \, d\theta \, d\varphi$$

$$= 4\pi r^2.$$



§ Vector fields on surfaces (do Carmo § 2.6)

<u>Def$^{\underline{n}}$</u>: A vector field on S is just a smooth map

$$V : S \longrightarrow \mathbb{R}^3$$

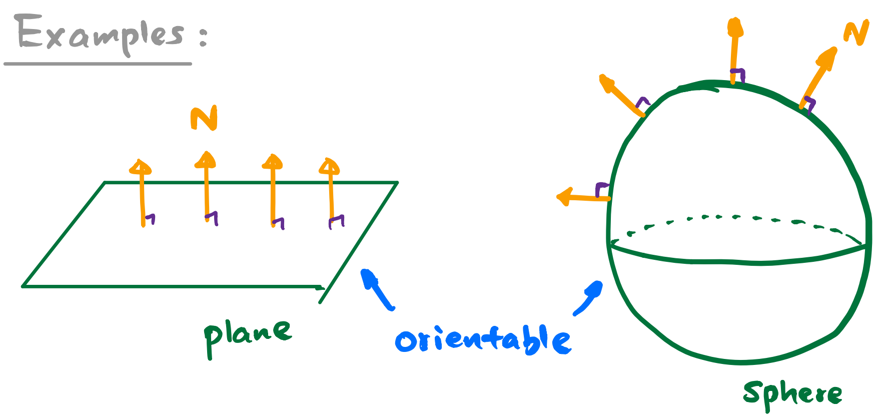
- V tangential if $V(p) \in T_pS \quad \forall p \in S$
- V normal if $V(p) \perp T_pS \quad \forall p \in S$

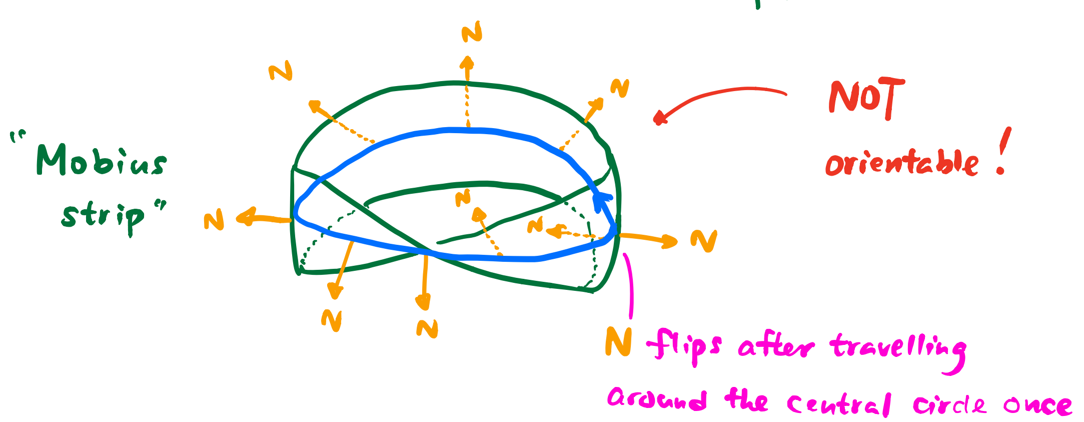
<u>Def$^{\underline{n}}$</u>: A surface $S \subseteq \mathbb{R}^3$ is orientable

if $\exists$ unit normal vector field $N : S \to \mathbb{R}^3$

i.e. $N(p) \perp T_pS$, $\|N(p)\| = 1 \quad \forall p \in S$

<u>Examples</u>:





Remarks: (1) Any orientable surface S has exactly 2 distinct orientations, N and $-N$

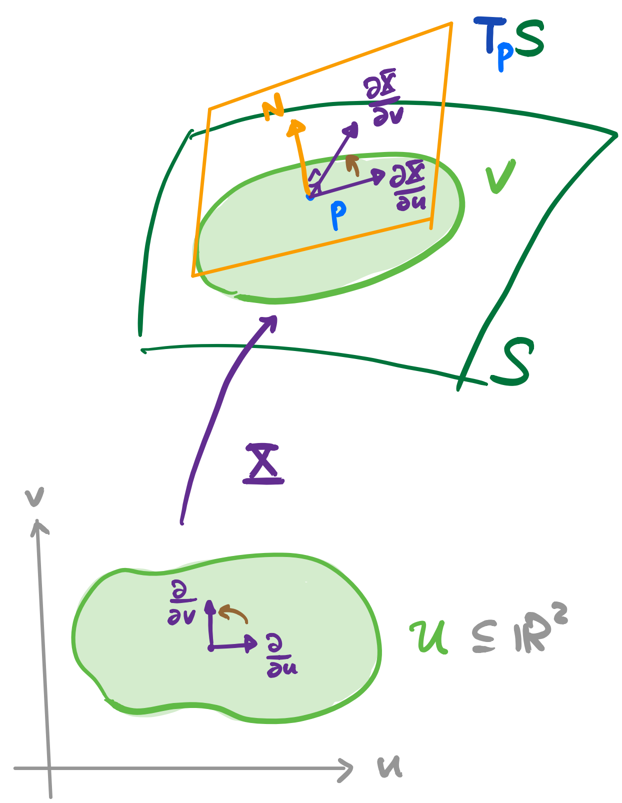
(2) Orientability is a global property.

FACT: Any surface is "locally orientable".

Fix a chart $\mathbf{X}: \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow V \subseteq S$

$$\boxed{N := \frac{\frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v}}{\left\| \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \right\|}}$$

unit normal vector field defined locally on V



Ex: Can you find another parametrization that induces the opposite orientation?

(3) One can also define orientability from an intrinsic point of view:

A smooth n-manifold M is orientable if $\exists$ collection of charts $\{X_\alpha : U_\alpha \subseteq \mathbb{R}^n \to M\}$ s.t. $\bigcup_\alpha X_\alpha(U_\alpha) = M$ and the transition maps

$$\psi_{\alpha\beta} = X_\beta^{-1} \circ X_\alpha : U_\alpha \xrightarrow{\cong} U_\beta$$

are orientation preserving diffeomorphisms.

Examples of orientable surfaces

(1) Graphical surfaces

$$S = \text{graph}(f) \quad \text{for } f : U \subseteq \mathbb{R}^2 \to \mathbb{R} \text{ smooth}$$

(2) Level surfaces

$$S = F^{-1}(a)$$

where $a \in \mathbb{R}$ is a regular value of $F : \mathbb{R}^3 \to \mathbb{R}$ smooth

Exercise: Prove that they are orientable surfaces.

§ Differential topology of surfaces

Def$^{\underline{n}}$: A surface $S \subseteq \mathbb{R}^3$ is closed if S is compact without boundary.

Classification of surfaces

A closed orientable surface is homeomorphic to one and only one of the following:

(*) 

genus (i.e. number of holes) :	0	1	2	…	g
Euler characteristic χ :	2	0	-2	…	$2-2g$

Remark: There is also a classification of non-orientable closed surfaces given by the real projective plane, Klein bottle, etc……